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An integral equation for unsteady surface waves and a comment on the Boussinesq equation

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An integral equation for unsteady inviscid surface waves has been obtained. Existing known approximations are all derived from the one equation. The Boussinesq equation is obtained and criticized.

1. Introduction

The study of time-dependent free-surface flows was first developed by Airy (1845) and Stokes (1849). The theory developed by Stokes was appropriate for small-amplitude waves of arbitrary wavelength. He showed that unsteady waves of this type were a superposition of steady progressive waves of the form

$$\eta = a \sin \alpha(x+ct). \tag{1.1}$$

The waves were dispersive with the wavelength $2\pi/\alpha$ and wave velocity c connected by the relation

$$\alpha c^2 = g \tanh \alpha h,$$

where g is the acceleration due to gravity and h is the undisturbed height.

Airy developed a theory for long waves of small amplitude. From the theory emerged the fact that long waves must change their form as they advance. However, this seemed to contradict the experimental discovery of the steady solitary wave by Russell (1844). Later Boussinesq showed that the incorrect predictions of the Airy theory were due to the assumption of infinite wavelength, or more specifically, due to the neglect of terms of order h^2/λ^2 (where λ is the wavelength). He used an unsteady theory and went on to develop what are now known as the Boussinesq equations. In terms of u the mean particle velocity and η the height above the undisturbed height, his equations take on the form

$$\frac{\partial \eta}{\partial t} + h\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(u\eta) = 0 \tag{1.2}$$

and

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} + \frac{h}{3}\frac{\partial^3 \eta}{\partial t^2\, \partial x} = 0. \tag{1.3}$$

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He also simplified these equations by eliminating u to get

$$\frac{\partial^2\eta}{\partial t^2}-gh\frac{\partial^2\eta}{\partial x^2} = gh\frac{\partial^2}{\partial x^2}\left(\frac{3\eta^2}{2}+\frac{h^2}{3}\frac{\partial^2\eta}{\partial x^2}\right). \tag{1.4}$$

Equation (1.4), however, has one disadvantage over (1.2) and (1.3), which to date seems to have gone unnoticed. Both equations describe waves moving in the positive and negative x direction with an approximate speed of $(gh)^{\frac{1}{2}}$ and both give η correct to the first order in a, the amplitude, However, (1.2) and (1.3) give the unsteady part (strictly speaking, the part that does not satisfy the wave equation) to order a^2, whereas (1.4) does not. This will be shown more thoroughly later.

Both sets of equations have steady solutions, one of which corresponds to the solitary wave, whose solution was obtained by Boussinesq. These permanent finite-amplitude wave forms were discovered by Korteweg & de Vries (1895), who called them cnoidal waves. These waves reduce the limiting case to the solitary wave found by Boussinesq. They also obtained a simpler equation for unsteady waves, that can be obtained from (1.2) and (1.3) by assuming that waves propagated only in one direction and by making an approximate integration. Their equation is

$$\eta_t+c(1+\tfrac{3}{2}(\eta/h)\,\eta_x)+\tfrac{1}{6}ch^2\eta_{xxx} = 0. \tag{1.5}$$

Both the Boussinesq and the Korteweg & de Vries equations have been studied extensively. Of the more recent work, Peregrine (1966) uses equations (1.2) and (1.3) for a numerical calculation of an undular bore. He also uses a modified version for the approach of long waves up a sloping beach. The numerical method used by Peregrine was extended by Chan, Street & Strelkoff (1969) to find the run up of a solitary wave on a wall. They also give a completely numerical solution of the Navier–Stokes equations for the case of the run up of a solitary wave. Mayer (1962) uses (1.4) in his discussion of the interaction of two solitary waves going in the opposite direction. Whitham (1965a, b, 1967) discusses solutions of both (1.4) and (1.5) and uses them to illustrate his theory of non-linear dispersive waves.

The main problem in obtaining an equation for η is that although the height depends on two variables, x and t, we have to introduce a third independent variable y, a vertical co-ordinate, which in a sense is not part of the (x,t) space in which the wave propagates. This is because the velocity components, whose values are needed on the free surface, depend on all three (x,y,t) variables. In this paper we will show that by a suitable transformation of co-ordinates we can reduce the equation to one where the variation with respect to the vertical co-ordinate is replaced by an integral over the free surface. Whitham (1967) makes the point that dispersion is best represented by an integral since all high-frequency effects are lost by the long wave expansion. The integral equations found here are of necessity very complicated since no approximation is made. However, it is hoped that from the exact equations we could obtain suitable solvable equations that would describe waves of greatest height with the Stokes 120° angle at the crest and also the alternative of breaking into bores. Both of these phenomena are essentially high-frequency effects rather than non-linear effects.

2. Derivation of the exact integral equation

We assume that the fluid is incompressible and inviscid and that it flows over a horizontal bottom. The motion is assumed to be two-dimensional and irrotational. The physical co-ordinate axes are chosen so that the bottom is $y = 0$, as shown in figure 1.

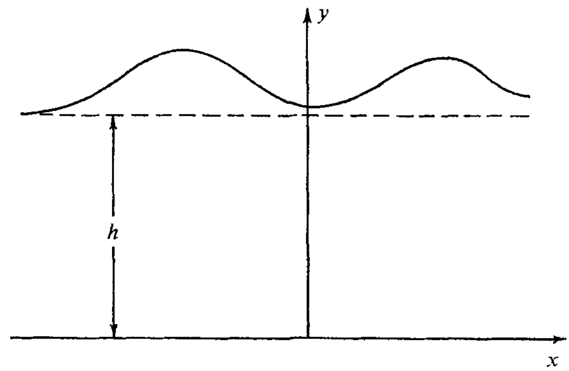

FIGURE 1. The co-ordinate axes.

The first step in the analysis is to transform from the physical co-ordinates (x, y, t) to new co-ordinates (ζ, ξ, τ) so that the flow region

$$-\infty \leqslant x \leqslant \infty, \quad 0 \leqslant y \leqslant h(x,t)$$

is transformed into the strip $-\infty \leqslant \zeta \leqslant \infty$, $0 \leqslant \xi \leqslant 1$. For completeness we also assume that $x = 0$ is transformed into $\zeta = 0$. This may be achieved by noticing that the reverse transformation is simply a Dirichlet problem with y given everywhere on the boundary $\eta = 0$ and $\eta = 1$.

Thus if $h(x,t) \Rightarrow h(\zeta, \tau)$ the transformation is given implicitly by (e.g. see Woods 1961)

$$y(\zeta, \xi, \tau) = \tfrac{1}{2} \sin \pi\xi \int_{-\infty}^{\infty} \frac{h(\zeta - \zeta_0, \tau)\, d\zeta_0}{\cos \pi\xi + \cosh \pi\zeta_0}, \tag{2.1}$$

$$x(\zeta, \xi, \tau) = \int_0^{\zeta} \frac{\partial y}{\partial \xi}(\zeta_0, \xi, \tau)\, d\zeta_0, \tag{2.2}$$

$$t = \tau. \tag{2.3}$$

We introduce a velocity potential ϕ by

$$\partial\phi/\partial x = u \quad \text{and} \quad \partial\phi/\partial y = v, \tag{2.4}$$

where u and v are the horizontal and vertical velocity components of the fluid.

Then the equations and boundary conditions that govern the solution of ϕ and h are

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0, \tag{2.5}$$

$$\partial\phi/\partial y = 0 \quad \text{on} \quad y = 0, \tag{2.6}$$

$$\frac{\partial h}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial h}{\partial x} - \frac{\partial \phi}{\partial y} = 0 \quad \text{on} \quad y = h(x,t). \tag{2.7}$$

A final equation is obtained from Bernoulli's equation, which states that

$$\frac{p}{\rho}+\frac{\partial\phi}{\partial t}+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+gy=f(t) \tag{2.8}$$

and in particular on the free surface, where the pressure is constant,

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial y}\right)^2+gh=f(t) \quad \text{on} \quad y=h(x,t). \tag{2.9}$$

Since the transformations (2.1) and (2.2) are conformal, in the sense that $x+iy$ is an analytic function of $\zeta+i\xi$, (2.5) is transformed simply to

$$\partial^2\phi/\partial\xi^2+\partial^2\phi/\partial\zeta^2=0 \tag{2.10}$$

and the boundary conditions (2.6) and (2.7) expressing the value of the normal derivative of ϕ are transformed into

$$\partial\phi/\partial\xi=0 \quad \text{on} \quad \xi=0, \tag{2.11}$$

$$\frac{\partial\phi}{\partial\xi}=\frac{\partial x}{\partial\xi}\frac{\partial y}{\partial\tau}-\frac{\partial x}{\partial\tau}\frac{\partial y}{\partial\zeta}=\frac{\partial(x,y)}{\partial(\zeta,\tau)} \quad \text{on} \quad \xi=1. \tag{2.12}$$

The problem of finding the function ϕ is now a Neumann problem since the normal derivative of ϕ is given on the boundaries $\xi=0$ and $\xi=1$ (again see Woods 1961).

Thus the solution for ϕ is

$$\phi(\zeta,\xi,\tau)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial(x,y)}{\partial(\zeta,\tau)}(\zeta-\zeta_0,\tau)\log(\cos\pi\xi+\cosh\pi\zeta_0)\,d\zeta_0. \tag{2.13}$$

The final equation that determines $h(\zeta,\tau)$ is given by substituting in (2.8). However, we first have to transform (2.8) into (ζ,ξ,τ) co-ordinates. The required form of (2.8) is

$$\frac{\partial\phi}{\partial\tau}+\left\{\frac{\partial\phi}{\partial\zeta}\frac{\partial(x,y)}{\partial(\xi,\tau)}-\frac{\partial\phi}{\partial\xi}\frac{\partial(x,y)}{\partial(\zeta,\tau)}\right\}\Big/\frac{\partial(x,y)}{\partial(\zeta,\xi)}$$
$$+\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial\zeta}\right)^2+\left(\frac{\partial\phi}{\partial\xi}\right)^2\right]\Big/\frac{\partial(x,y)}{\partial(\zeta,\xi)}+g(h)=f(t) \quad \text{on} \quad \xi=1. \tag{2.14}$$

All the derivatives in the ξ direction can be obtained as integrals over the whole range of ζ. Thus we have eliminated the ξ dependence and are left with an integral equation where ζ and τ are the only independent variables.

3. Derivation of existing theories from the integral equation

By using the appropriate approximations we can derive existing theories for unsteady surface waves.

Stokes waves

The Stokes wave is derived by assuming that the equation for the height η above the mean water level can be linearized in η.

We have to exercise care when we are dealing with waves that do not tend to zero as $x \to \infty$. The kernel in the integrand of (2.13) tends to $|\zeta_0| + \text{constant}$ as $\zeta_0 \to \infty$ and if $\partial(x,y)/\partial(\zeta,\tau)$ is sinusoidal then we must take the generalized value of the integral. For a more careful argument see Lighthill (1958).

The solution of (2.13) and (2.14) is most easily obtained by taking Fourier transforms with respect to ζ along $\xi = 1$. If we denote the Fourier transform with respect to ζ by bars, then from (2.13)

$$\begin{aligned}\bar{\phi}(k,\tau) &= (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{ik\zeta}\phi(\zeta,1,\tau)\,d\zeta \\ &= -(2\pi)^{-\frac{3}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik\zeta}X(\zeta-\zeta_0,\tau)\log(\cosh\pi\zeta_0-1)\,d\zeta_0 \\ &= -(2\pi)^{-\frac{1}{2}}\bar{X}(k,\tau)\,\overline{\log(\cosh\pi\zeta-1)} \\ &= -k^{-1}\coth k\,\bar{X}(k,\tau),\end{aligned} \tag{3.1}$$

where
$$X = \frac{\partial x}{\partial\zeta}\frac{\partial y}{\partial\tau} - \frac{\partial x}{\partial\tau}\frac{\partial y}{\partial\zeta}. \tag{3.2}$$

Then if
$$y = h_0 + \eta(\zeta,\tau) \tag{3.3}$$

to a first approximation
$$x = h_0\,\partial\eta/\partial\tau, \tag{3.4}$$

so that (3.1) becomes
$$\bar{\phi} = -k^{-1}\coth k\,\overline{\partial\eta}/\partial\tau. \tag{3.5}$$

Then linearization of (2.14) yields

$$\phi_\tau + g\eta = f(\tau) - f(h_0). \tag{3.6}$$

Again taking the generalized Fourier transforms, we obtain

$$\bar{\phi}_\tau + g\bar{\eta} = 0. \tag{3.7}$$

Thus by eliminating ϕ we obtain

$$g\bar{\eta} + h_0 k^{-1}\coth k\,\partial^2\bar{\eta}/\partial\tau^2 = 0, \tag{3.8}$$

with solution
$$\bar{\eta} = A(k)\sin(pt + \alpha(k)), \tag{3.9}$$

where
$$p^2 = gk\tan k/h_0. \tag{3.10}$$

Thus
$$\eta = \int_{-\infty}^{\infty} B(k)\sin(pt + k\zeta + \alpha(k))\,dk \tag{3.11}$$

with the usual dispersion relation

$$p^2 = gk\tanh K/h_0. \tag{3.12}$$

The zeroth approximation of (2.2) suffices to determine x so that

$$x = \int_0^\zeta (\partial y/\partial\xi)\,d\zeta = h_0\zeta \tag{3.13}$$

and
$$\eta = \int_{-\infty}^{\infty} B(k)\sin[(pt + (kx/h_0)) + \alpha k]\,dk. \tag{3.14}$$

The Boussinesq equation

We can now recover the fourth-order partial differential equation corresponding to (1.4) by making the assumption that N, the non-dimensionalized height above the mean water level, is of order a and that $\partial^{i+j}N/\partial\zeta^j\partial\tau^i = O(a^{\frac{1}{2}(i+j)+1})$ and retaining terms up to order a^3. Again as in the case of Stokes waves we have to be careful when we are studying waves that do not tend to zero at ∞. For example, we might have uniform cnoidal waves at infinity or we might have a bore situation where the water levels at $\pm\infty$ are different. In these problems difficulties arise because we have to specify the velocity at ∞. These difficulties can be overcome by differentiation. Thus if we differentiate (2.13) twice with respect to ζ then the kernel becomes exponentially small at ∞ and so the integral converges. Then we can satisfy the conditions at ∞ by integrating or simply leave the conditions as initial conditions to be satisfied by the solution of the final differential equation. Similarly, the function $f(\tau)$ can be eliminated by one differentiation with respect to ζ. Here we will not deal with these problems but restrict ourselves to flows where N tends to zero at ∞.

To non-dimensionalize (2.13) and (2.14) we use $(gh_0)^{\frac{1}{2}}$ as a velocity scale and h_0 as a length scale. Denoting non-dimensional quantities by capitals we can obtain, after lengthy but straightforward calculations,

$$\Phi(\zeta,\tau) = \int_0^{\zeta}\int_{\zeta_1}^{\infty}\left\{[1+N(\zeta_2,T)]\,N_T(\zeta_2,T) + N_3(\zeta_2,T)\int_0^{\zeta_2}N_T(\zeta_3,T)\,d\zeta_3\right\}d\zeta_2\,d\zeta_1+\tfrac{1}{3}N_T+\ldots, \tag{3.15}$$

and
$$\Phi_T+\frac{3}{2}\left[\int_0^{\zeta}N_T(\zeta_1,T)\,d\zeta_1\right]^2-2\int_0^{\zeta}N_T(\zeta_1,T)\,d\zeta_1+\frac{1}{2}\left[\int_0^{\infty}N_T(\zeta_1,T)\,d\zeta_1\right]^2 + N+\ldots = F(T)-1. \tag{3.16}$$

Elimination of Φ then gives, after two differentiations with respect to ζ,

$$\tfrac{1}{3}N_{\zeta\zeta TT}+N_{\zeta\zeta}-N_{TT}-\tfrac{1}{2}(N^2)_{TT}+4N_{T\zeta}\int_0^{\zeta}N_T+N_{\zeta}\int_0^{\zeta}N_{TT}+3(N_T)^2 - 2N_{T\zeta}\int_0^{\infty}N_T(\zeta_1,T)\,d\zeta_1 = 0. \tag{3.17}$$

This is the appropriate form of the Boussinesq equation in (ζ, T) variables. If we revert to (X,T) variables we can obtain

$$\tfrac{1}{3}N_{XXTT}+N_{XX}-N_{TT}+2NN_{XX}+N_X^2+2N_T^2-NN_{TT}-2N_{TX}\int_X^{\infty}N_T = 0. \tag{3.18}$$

The steady solitary wave solution can be found by looking for a solution of the form

$$N = aN_1(X+FT). \tag{3.19}$$

We then find that

$$N_1 = (F^2-1)/(aF^2)\,\mathrm{sech}^2\,\tfrac{1}{2}[3(F^2-1)]^{\frac{1}{2}}\,(X+FT). \tag{3.20}$$

Thus $F^2 = 1+a$ to order a^2 and we can write

$$N_1 = \mathrm{sech}^2\,\tfrac{1}{2}(3a)^{\frac{1}{2}}\,(X+FT). \tag{3.21}$$

It should be noted that (1.4) and (3.18) are different unless we replace $\partial/\partial T$ by $\partial/\partial X$ in the higher-order terms. This is permissible to our order of accuracy in the terms NN_{TT} but not in the terms N_T^2. The difference between the two equations is that (3.18) includes all terms that give a contribution of order a^3, whereas (1.4) represents an equation that gives N correct to order a. Thus while (1.4) and (3.18) give aN_1, the first approximation to N, accurately, (3.18) gives us partial information about the second approximation.

For example, if we look for a solution appropriate to two waves going in opposite directions and substitute

$$N = aN_1(X+F_1T)+aN_2(X-F_2T), \tag{3.22}$$

we find that the equation is not separable unless we introduce a term a^2N_3 to the right-hand side of (3.22). This gives an order a^3 term in (3.18), and we obtain

$$\begin{aligned}\sum_{i=1}^{2} (F_i^2/(3a)\,N_i^{\text{iv}} - (F_i^2-1)/aN_i'' + (2+F_i^2)\,N_iN_i'' + N_i'^2(1+2F_i^2)) \\ = N_{3TT}-N_{3XX}-N_1''N_2(2-2F_1F_2-F_1^2)-N_2''N_1(2-F_1F_2-F_2^2) \\ +N_1'N_2'(4F_1F_2-2).\end{aligned} \tag{3.23}$$

Again to order a^2 we can write

$$F_i^2 = 1+k_ia, \tag{3.24}$$

where k_i are constants of order one.

Then, to complete the separation of variables we must make the right-hand side of (3.23) zero by choosing

$$N_3 = \frac{(2F_2F_1-1)}{2(1+F_1F_2)}\left[N_2'\int_0^{X+FT} N_1(\xi)\,d\xi + N_1'\int_0^{X-FT} N_2(\xi)\,d\xi + 2N_1N_2\right]. \tag{3.25}$$

Then we can obtain the solution for N_1 and N_2 as

$$N_i = K_i\,\text{sech}^2\,\tfrac{1}{2}(3aK_i)^{\frac{1}{2}}\,(X-(-1)^iF_iT) \tag{3.26}$$

and provided F_1 and F_2 are the same sign

$$\begin{aligned}N_3 = {} & \tfrac{1}{2}K_1K_2\,\text{sech}^2\,\tfrac{1}{2}\,(3aK_1)^{\frac{1}{2}}\,(X+F_1T)\,\text{sech}^2\,\tfrac{1}{2}\,(3aK_2)^{\frac{1}{2}}\,(X-F_2T) \\ & -\tfrac{1}{2}K_1K_2[(K_1/K_2)^{\frac{1}{2}}\,\text{sech}^2\,\tfrac{1}{2}(3aK_i)^{\frac{1}{2}}\,(X+F_1T)+(K_2/K_1)^{\frac{1}{2}}\,\text{sech}^2\,\tfrac{1}{2}(3aK_2)^{\frac{1}{2}} \\ & \times(X-F_2T)]\times\tanh\tfrac{1}{2}(3aK_1)^{\frac{1}{2}}\,(X+F_1T)\tanh\tfrac{1}{2}(3aK_2)^{\frac{1}{2}}\,(X-F_2T) \\ & +N_4(X+F_1T)+N_5(X-F_2T).\end{aligned} \tag{3.27}$$

If we use the already-known second approximations to the steady solitary wave (Laitone 1960) we can obtain N_4 and N_5. Thus the complete second approximation to the interaction of two solitary waves is

$$\begin{aligned}N = \sum_{i=1}^{2}\{aN_i-\tfrac{3}{4}a^2N_i(K_i-N_i)\}+\tfrac{1}{2}a^2N_1N_2-\tfrac{1}{2}a^2\,(K_1K_2)^{\frac{1}{2}}\,(N_1+N_2) \\ \times\tanh\tfrac{1}{2}(3\alpha_1)^{\frac{1}{2}}\,(X+F_1T)\tanh\tfrac{1}{2}(3\alpha_2)^{\frac{1}{2}}\,(X-F_2T),\end{aligned} \tag{3.28}$$

where
$$N_i = K_i\,\text{sech}^2\,\tfrac{1}{2}(3\alpha_i)^{\frac{1}{2}}\,(X-(-1)^iF_iT), \tag{3.29}$$

$$F_i^2 = 1+K_ia-K_i^2a^2/20+O(a^3), \tag{3.30}$$

and
$$\alpha_i = k_ia(1-\tfrac{5}{4}K_ia)+O(a^3). \tag{3.31}$$

The run up, R, of a solitary wave of amplitude, a, against a vertical wall can be obtained by putting $F_1 = F_2$ and $K_1 = K_2 = 1$ and finding N at $X = T = 0$. This gives

$$R = 2a + \tfrac{1}{2}a^2 + O(a^3). \tag{3.32}$$

This compares favourably with the experiments of Camfield & Street (1969) and the numerical results of Chan, Street & Strelkoff (1969) (see figure 2). Equation (3.28) does not agree with the interaction obtained by Benney & Luke (1964). However, their result has to be in error since the travelling wave part of the solution does not agree with the second approximation of Laitone (1960).

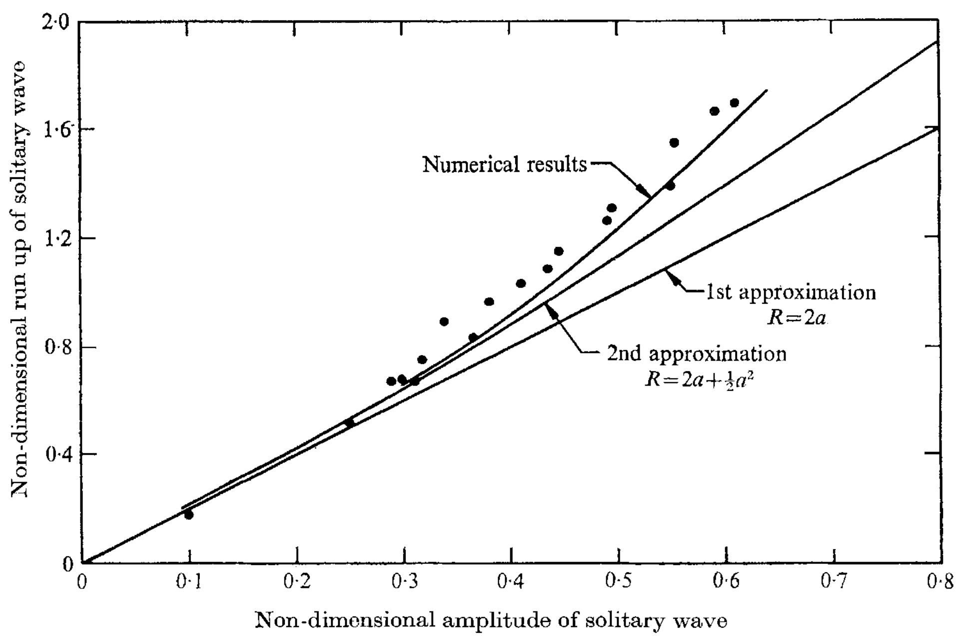

FIGURE 2. Run up of the solitary wave against a vertical wall plotted as a function of wave amplitude. Figure also includes numerical results of Chan, Street & Strelkoff (1969) and the experimental results of Chamfield & Street (1968).

This approach also shows that two solitary waves of almost equal speeds going in the same direction must have an order a interaction. This is because the term $(1 + F_1F_2)$ in the denominator of expression (3.25) becomes zero to order a. Thus we are unable to make the right-hand side of (3.23) zero with a term a^2N_3. This poses many questions as to what happens in this interesting case. Does the interaction rise to order a and fall to zero as the waves pass through each other, or do the two waves coalesce and form one wave with a different wave velocity and amplitude?

The work done by Lax (1968) suggests that two solitary waves travelling in the same direction do emerge unaltered, except for a possible change in phase. However, he uses only the Korteweg & de Vries equation and does not produce an analytic solution.

4. Conclusions

An integral equation has been derived for the unsteady motion of the free surface of an inviscid fluid. It has been shown that existing theories can be obtained from the integral equation by making the appropriate assumptions. The correctly derived Boussinesq equation is found to contain the interaction of two steady wave trains going in opposite directions. This interaction is found for the case of two solitary waves. When the waves are equal the result has been compared with available experimental observations and numerical results for the run up of a solitary wave against a vertical wall.

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